MISSILE DYNAMICS

INTRODUCTION OF THE ANGLE OF SIDESLIP

In the course of this lecture, we will have to abandon the restriction to flight in a vertical plane. The vector of the ballistic wind will no longer be restricted to the plane of symmetry of the missile. Therefore, it becomes necessary to introduce a new angle, called the angle of sideslip, β , for measuring the deviation of the velocity vector from the plane of symmetry.

Figure 9 shows three coordinate systems:

x, y, z are axes rigidly connected with the missile

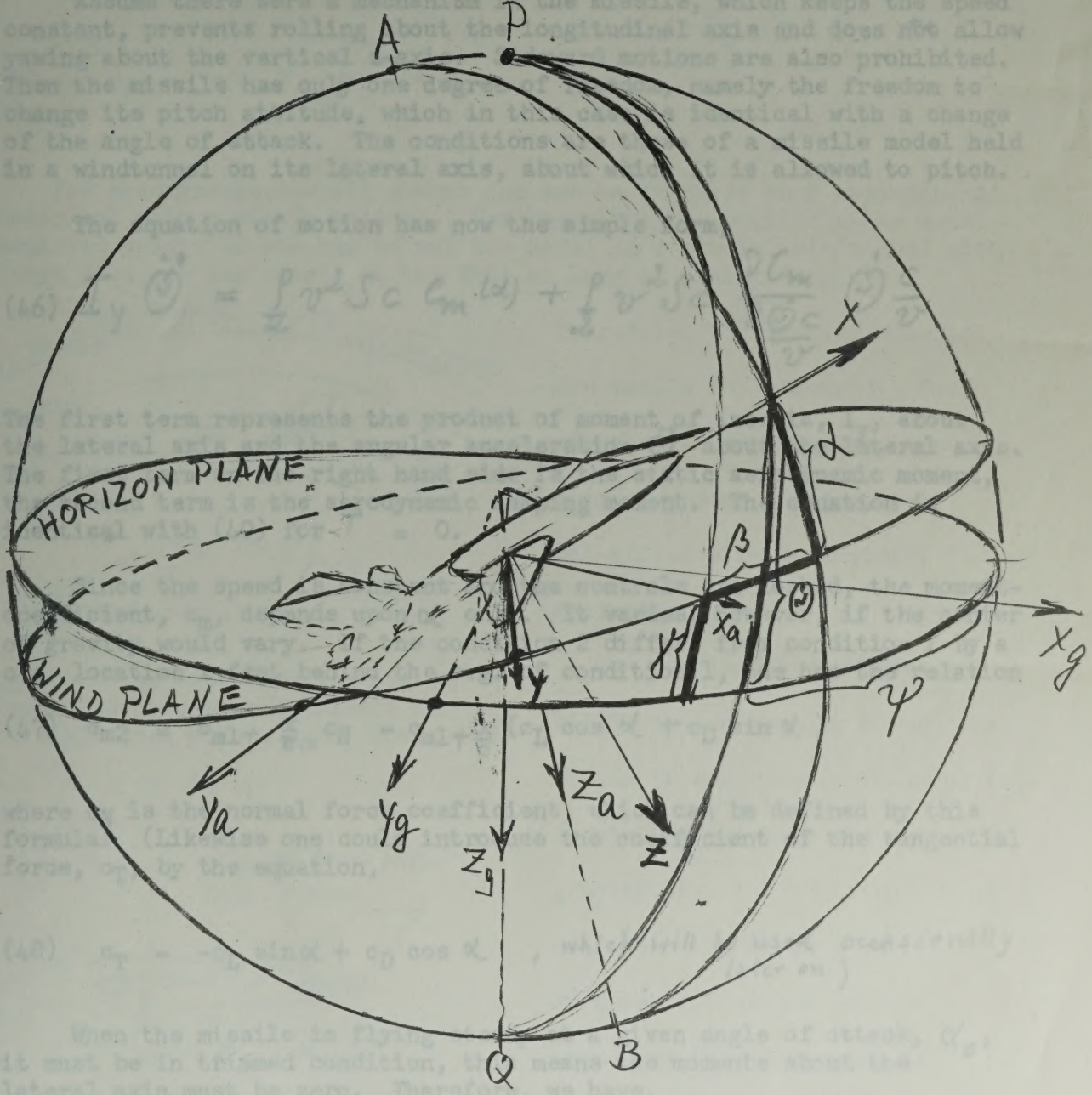
x_g, y_g, z_g are axes fixed with the ground

x_a, y_a, z_a, are axes fixed to the ballistic wind, which is assumed to be given by the tangent of the trajectory (no natural wind)

Besides the wind vector the wind axes need an additional determination. The $x_a y_a$ plane is so defined that it contains the lateral axes, y, of the missile. Unfortunately one needs all three coordinate systems: The mass forces originate from magnitudes given in the x,y,z system, the aerodynamic forces are given by α and β (this means they are connected with the x_a, y_a, z_a system), the gravitational forces are fixed to the earth system x_g, y_g, z_g.

MOTION OF THE MISSILE WITH LOCKED CONTROLS

We will restrict our discussions to the simplest case, the motion of the missile with locked controls. The most general case of this type of motion can be built up of seven elements, the seven modes of missile motion. We will discuss these modes as if they would exist independently from each other. We will, however, mention how the characteristics of these modes, i.e., the frequency and the damping of the modes, are being changed by the presence of the other modes.



α = angle of attack

β = angle of sideslip

x,y,z missile fixed system

x_a, y_a, z_a system of wind axes

x_g, y_g, z_g axes fixed to the earth

Figure 9. The three coordinate systems of missile dynamics.

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THE FAST PITCHING MODE

Assume there were a mechanism in the missile, which keeps the speed constant, prevents rolling about the longitudinal axis and does not allow yawing about the vertical z-axis. Sideward motions are also prohibited. Then the missile has only one degree of freedom, namely the freedom to change its pitch attitude, which in this case is identical with a change of the angle of attack. The conditions are those of a missile model held in a windtunnel on its lateral axis, about which it is allowed to pitch.

The equation of motion has now the simple form,

$$(46)\quad I_y \ddot{\vartheta} = \frac{\rho}{2} v^2 S c\, c_m(\alpha) + \frac{\rho}{2} v^2 S c \frac{\partial c_m}{\partial \frac{\dot{\vartheta} c}{v}} \dot{\vartheta} \frac{c}{v}$$

The first term represents the product of moment of inertia, I_y, about the lateral axis and the angular acceleration $\ddot{\vartheta}$ about the lateral axis. The first term on the right hand side is the static aerodynamic moment, the second term is the aerodynamic damping moment. The equation is identical with (40) for $T = 0$.

Since the speed is constant and the controls are locked, the moment-coefficient, c_m, depends upon α only. It varies, however, if the center of gravity would vary. If the condition 2 differs from condition 1 by a c.g. location x feet behind the c.g. of condition 1, one has the relation

$$(47)\quad c_{m2} = c_{m1} + \frac{x}{c} c_N = c_{m1} + \frac{x}{c}(c_L \cos\alpha + c_D \sin\alpha)$$

where c_N is the normal force coefficient, which can be defined by this formula. (Likewise one could introduce the coefficient of the tangential force, c_T, by the equation,

$$(48)\quad c_T = -c_L \sin\alpha + c_D \cos\alpha \quad ,$$ which will be used occasionally later on)

When the missile is flying steady at a given angle of attack, α_o, it must be in trimmed condition, this means the moments about the lateral axis must be zero. Therefore, we have,

$$(49)\quad c_m(\alpha_o) = 0$$

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The equation of motion has now the simple form,

$$I_y \ddot{\vartheta} = \frac{1}{2}\rho v^2 S c \, c_m(\alpha) + \frac{1}{4}\rho v^2 S c^2 \frac{\partial c_m}{\partial \frac{\dot{\vartheta} c}{v}} \frac{\dot{\vartheta} c}{v} \qquad (46)$$

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We discuss now, what motion the missile will perform, it its angle of attack is changed by a disturbance, $\Delta\alpha$, as for instance a gust. If the disturbance is small, that is if $\Delta\alpha$ is small, one can write on account of (49),

$$(50) \qquad c_m = 0 + \frac{dC_m}{d\alpha}\Delta\alpha$$

The aerodynamic damping moment can not be measured in a conventional windtunnel with a static balance. One has either to oscillate the model about its y-axis or one has to put the model in a special windtunnel with curved flow, as one exists at the NACA at Langley Field, Virginia.

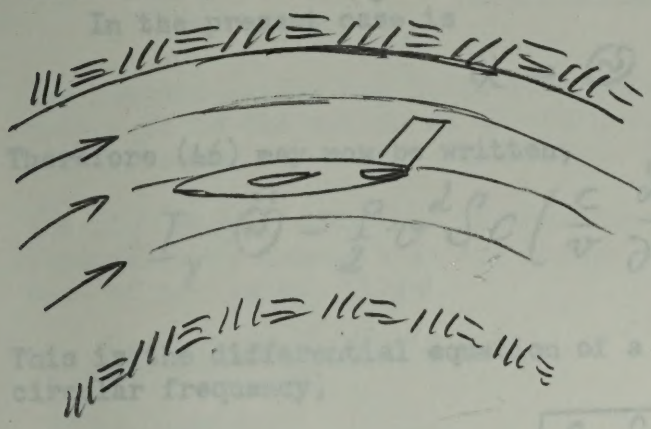

Fig. 10. The pitch damping may be measured by a static balance in a windtunnel with curved flow.

Therefore, the damping moments are mostly not measured. For missiles with tailplanes the damping moment can easily be calculated with sufficieint accuracy.

The fuselage contributes so little to the damping moment that it can be neglected. The wing is located so close to the center of gravity that its contribution can be taken care of as a correction of about 10% of the damping caused by the horizontal tail plane, which gives the only important part.

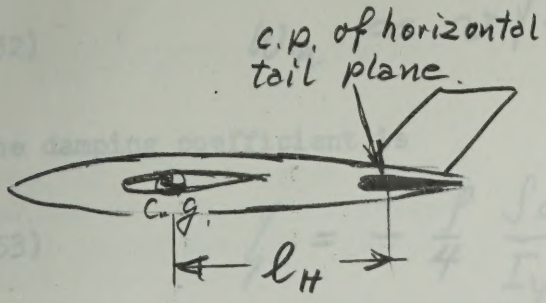

Fig. 11. $\dot{\vartheta}$ causes a change of angle of attack on the tailplane, which produces the damping moment.

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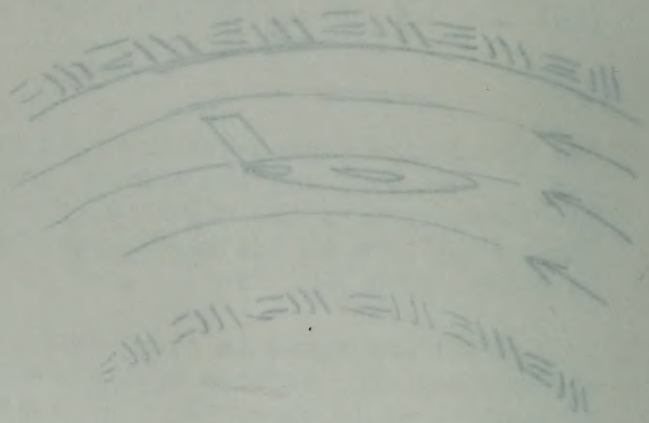

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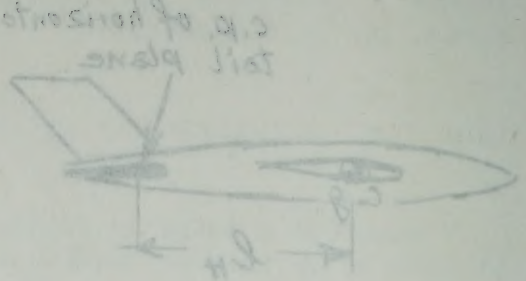

Fig. 11. Causes a change of angle of attack on the tailplane, which produces the damping moment.

If l_H is the distance of the horizontal tail plane from the center of gravity, one has in the center of pressure of the tailplane a change

of angle of attack due to pitch of the amount $l_H \dot{\omega}/v$. If S_H is the area of the horizontal tail plane and if $\left(\frac{dC_L}{d\alpha}\right)_H$ is the slope of its lift coefficient, one obtains the damping moment

$$M_D = -\frac{\rho}{2} v^2 S_H \left(\frac{dC_L}{d\alpha}\right)_H \frac{l_H \dot{\omega}}{v}$$

Consequently one obtains by comparison with the expression for the damping moment in (46),

$$\frac{\partial C_m}{\partial \frac{\dot{\omega} c}{v}} = -\frac{S_H}{S} \frac{l_H^2}{c^2} \left(\frac{dC_L}{d\alpha}\right)_H \tag{50}$$

In the present case is

$$\alpha = \omega$$

Therefore (46) may now be written,

$$I_y \ddot{\omega} - \frac{\rho}{2} v^2 S c \left(\frac{c}{v} \frac{\partial C_m}{\partial \frac{\dot{\omega} c}{v}} \dot{\omega} + \frac{\partial C_m}{\partial \alpha} \omega \right) = 0$$

This is the differential equation of a damped oscillation with the circular frequency,

$$w_n = v \sqrt{\frac{\rho}{2} \frac{Sc}{I_y}} \sqrt{-\frac{\partial C_m}{\partial \alpha} - \frac{\rho}{2} \frac{Sc^3}{I_y} \left(\frac{\partial C_m}{\partial \frac{\dot{\omega} c}{v}}\right)^2} \tag{51}$$

Which is in very good approximation

$$w_n \approx v \sqrt{-\frac{\rho}{2} \frac{Sc}{I_y} \frac{\partial C_m}{\partial \alpha}} \tag{52}$$

The damping coefficient is

$$\zeta = -\frac{\rho}{4} \frac{Sc^2}{I_y} v \frac{\partial C_m}{\partial \frac{\dot{\omega} c}{v}} \tag{53}$$

These formulae show:

An oscillatory motion exists only if $\frac{\partial C_m}{\partial \alpha}$ is negative. This was to be expected, because negative $\frac{\partial C_m}{\partial \alpha}$ means that an increase of the angle of attack causes a moment, which tries to reduce this increase; it means a restoring moment. A positive $\frac{\partial C_m}{\partial \alpha}$ would result in a divergent motion.

If the missile has a horizontal tailplane, the damping constant of the pitching mode is always positive. There have been measured only a certain type of delta wings, for which a narrow range of Mach numbers around one exists, in which these wings, when used without tailplane, would have a negative damping factor. A negative damping factor, of course, has to be avoided because it would result in an increase of disturbances.

The pitching mode couples so little with the other modes that the equations (52) and (53) can also be used, if other modes are present.

THE PHUGOID MOTION

The phugoid motion has first been treated by Lancaster who coined the name. It is the motion an aircraft would perform, if it had a very high value of the static stability. ($-\frac{\partial C_m}{\partial \alpha} \gg 1$). The effect of the high static stability is that the missile keeps its angle of attack constant. We assume that the thrust balances the drag and discuss the case of horizontal flight.

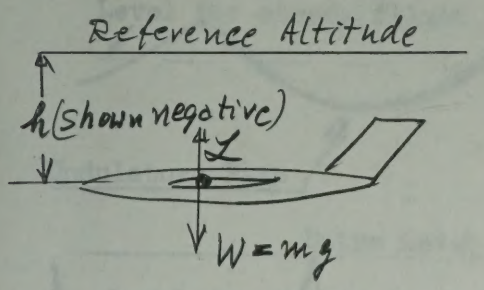

Fig. 12. The phugoid is a motion of exchange of potential energy (into kinetic energy) and vice versa.

The equation of the vertical motion is

$$(54)\quad m\frac{d^2h}{dt^2} = L - mg = \frac{\rho}{2}v^2 C_L S - mg$$

Since there is no energy dispersion (drag balanced by thrust), the sum of potential and kinetic energy must remain constant:

$$m\,(g \cdot h + v^2/2) = \text{Constant}.$$

If we count h from such an altitude, where v becomes zero, the right hand side of the equation is zero and one has

$$(55)\quad v^2 = -2gh$$

Substituting (55) into (54) one obtains

$$(56)\quad m\frac{d^2h}{dt^2} + \rho C_L g h = 0$$

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$$m\frac{d^2h}{dt^2} + \rho C_L g h = 0 \tag{56}$$

The lift coefficient c_L, can be expressed by the speed, v_o, of of steady horizontal flight with the lift coefficient and (56) is transformed into

$$c_L = \frac{2mg}{\rho v_o^2 S}$$

$$(57) \quad \frac{d^2h}{dt^2} + \frac{2g^2}{v_o^2} h = 0.$$

This is a harmonic oscillation with the circular frequency

$$(58) \quad \omega_p = \frac{g}{v_o}\sqrt{2} \quad (\text{sec}^{-1}) \qquad \text{or the period}$$

$$(59) \quad P_p = \pi\sqrt{2}\,\frac{v_o}{g} \quad (\text{sec})$$

The phugoid trajectories can be represented by mathematical expressions in closed form. There exist three different types depending upon the magnitude of the initial deviation from the steady state condition: the undulatory path, the cusped path and the looped path (see Fig. 13). It has to be mentioned that the phugoid investigations are not restricted to small disturbances, as most of the other disturbance investigations.

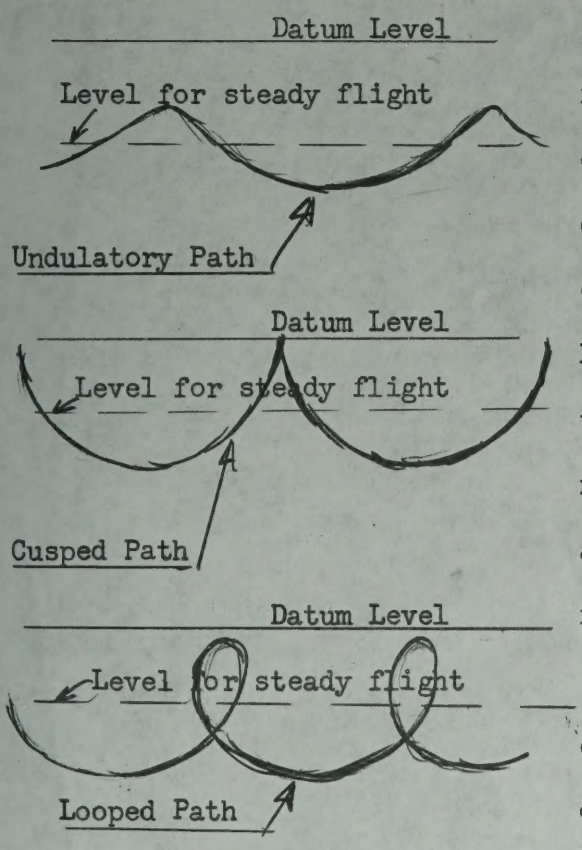

Fig. 13. The three types of Phugoid Motion.

If one would perform a more extended investigation and include other degrees of freedom and especially drag variations, one would obtain for the period and the damping of the phugoid mode,

$$(60) \quad P_r = \frac{\pi\sqrt{2}}{\sqrt{1 + \frac{M}{2C_L}\frac{\partial C_L}{\partial M}}}\,\frac{v_o}{g} \quad [\text{sec}]$$

The lift coefficient C_L can be expressed by the speed, V_o, of steady horizontal flight with the lift coefficient and (56) is transformed into

$$C_L = \frac{2W}{\rho S V_o^2}$$

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